Theorem: There exists an absolute constant $c>0$ such that if χ a character mod q, then the region $R_q = \left\{ s : Re(s) > 1 - \frac{c}{\log(q(|Im(s)|+1))} \right\}$ contains no zero of $L(s,\chi)$, unless χ is quadratic, in which case $L(s,\chi)$ has at most one, necesarilly real and simple zero in R_q.

Proof: If $\chi \pmod q$ is induced by $\chi^* \pmod{q^*}$,

then $L(s,\chi) = L(s,\chi^*) \prod_{p|q} \left(1 - \frac{\chi^*(p)}{p^s}\right)$.

This implies zeros of $L(s,\chi)$ and $L(s,\chi^*)$ coincide in $Re(s)>0$

$$\left(1 - \frac{\chi^*(p)}{p^s} = 0 \iff s = \frac{\underbrace{\log \chi^*(p) + 2\pi i n}_{\in i\mathbb{R}}}{\log p}\right)$$

So we may assume WLOG χ is primitive.

In particular, we may assume $\chi \neq \chi_0$ (since we know zero-free region for $\zeta(s)$)

Note that, for $\sigma > 1$, similarly as before,

$$\mathrm{Re}\left[-3\frac{L'}{L}(\sigma,\chi_0)-4\frac{L'}{L}(\sigma+it,\chi)-\frac{L'}{L}(\sigma+2it,\chi^2)\right]$$

$$=\sum_{(n,q)=1}\frac{\Lambda(n)}{n^\sigma}\left(3+4\,\mathrm{Re}(\chi(n)n^{-it})+\mathrm{Re}(\chi(n)^2n^{-2it})\right)$$

$$\left(\chi(n)\,n^{-it}=e^{i\theta_n}\right)$$

$$=\sum_{(n,q)=1}\frac{\Lambda(n)}{n^\sigma}\left(3+4\cos\theta_n+\cos(2\theta_n)\right)\ge 0.$$

We mimic the proof of zero-free region for $\zeta(s)$.

Let $1<\sigma\le 2$.

$$-\frac{L'}{L}(\sigma,\chi_0)=\sum_{(n,q)=1}\frac{\Lambda(n)}{n^\sigma}\le-\frac{\zeta'}{\zeta}(\sigma)=\frac{1}{\sigma-1}+O(1).$$

Let $\rho=\beta+it$ be a zero with $\beta>1-\dfrac{\delta}{\log(q(|t|+2))}$

$$\mathrm{Re}\left(-\frac{L'}{L}(\sigma+it,\chi)\right)\le-\mathrm{Re}\left(\frac{1}{s-\rho}\right)+O(\log(q(|t|+2))).$$

$$=-\frac{1}{\sigma-\beta}+O(\log(q(|t|+2)))$$

<u>Case 1</u>: χ complex, so $\chi^2\ne\chi_0$.

The character χ^2 may not be primitive

Say χ^2 is induced from χ^* modulo $2\le q^*\le q$ (since $\chi^2\ne\chi_0$, this is crucial!!).

For $\mathrm{Re}(s) > 1$,

$$\frac{L'}{L}(s,\chi^2) - \frac{L'}{L}(s,\chi^*) = \sum_{p \mid q} \frac{d}{ds}\left(1 - \frac{\chi^*(p)}{p^s}\right)$$

$$= \sum_{p \mid q} \frac{\chi^*(p) p^{-s} \log p}{1 - \chi^*(p) p^{-s}} \ll \sum_{p \mid q} \log p \le \log q$$

Hence $\mathrm{Re}\left(-\frac{L'}{L}(\sigma + 2it, \chi^2)\right) = -\mathrm{Re}\left(-\frac{L'}{L}(\sigma + 2it, \chi^*)\right)$

$+ O(\log q)$.

Using partial fraction expansion of $\frac{L'}{L}(s,\chi^*)$,

$\mathrm{Re}\left(-\frac{L'}{L}(\sigma + 2it, \chi^*)\right) \le C \cdot \log(q(|t|+2))$,

for some universal $C > 0$.

Therefore, putting it all together:

$$\frac{4}{\sigma - \beta} \le \frac{3}{\sigma - 1} + O(\log(q(|t|+2))).$$

Choose $\sigma = 1 + \frac{4\delta}{\log(q(|t|+2))}$

Then $4\frac{\log(q(|t|+2))}{5\delta} \le 3\frac{\log(q(|t|+2))}{4\delta} + O(\log(q(|t|+2)))$

Contradiction for δ small enough.

Case 2: χ quadratic, $\chi^2 = \chi_0$

Since $L(s,\chi_0) = \zeta(s) \prod_{p|q}\left(1-\frac{1}{p^s}\right)$, then for $\mathrm{Re}(s) > 1$, we have $\frac{L'(s,\chi_0)}{L(s,\chi_0)} - \frac{\zeta'}{\zeta}(s) \ll \log q$

(same argument as in previous case)

Hence $\mathrm{Re}\left(-\frac{L'}{L}(\sigma+2it,\chi^2)\right) = -\mathrm{Re}\left(\frac{\zeta'}{\zeta}(\sigma+2it)\right) + O(\log q)$

$$\leq \mathrm{Re}\left(\frac{1}{\sigma-1+2it}\right) + O(\log(q(|t|+2))).$$

(using same arguments as before and partial fractional expansion for $\frac{\zeta'}{\zeta}(s)$, where the term $-\frac{1}{s-1}$ appears).

Case 2.1 $|t| \geq \frac{\delta}{\log q}$. Choose $\sigma = 1 + \frac{4\delta}{\log(q(|t|+2))}$

Then $\mathrm{Re}\left(\frac{1}{\sigma-1+2it}\right) = \frac{\sigma-1}{(\sigma-1)^2+(4t)^2} \ll \log q.$

So we have $\frac{4}{\sigma-\beta} \leq \frac{3}{\sigma-1} + O(\log(q(|t|+2)))$.

Obtain same contradiction as before ✓

__Case 2.2__: $0 < |t| < \frac{\delta}{\log q}$.

Since χ is real, if ρ is a zero of $L(s,\chi)$, $\bar{\rho}$ is also a zero. Hence

$$-\frac{L'}{L}(\sigma,\chi) \le -\frac{1}{\sigma-\beta-it} - \frac{1}{\sigma-\beta+it} + O(\log q)$$

$$= -\frac{2(\sigma-\beta)}{(\sigma-\beta)^2+t^2} + O(\log q).$$

On the other hand,

$$-\frac{L'}{L}(\sigma,\chi) = \sum_n \frac{\Lambda(n)\chi(n)}{n^\sigma} \ge -\sum_n \frac{\Lambda(n)}{n^\sigma} = \frac{\zeta'}{\zeta}(\sigma)$$

$$\ge -\frac{1}{\sigma-1} + O(1).$$

$$\Longrightarrow -\frac{1}{\sigma-1} \le -\frac{2(\sigma-\beta)}{(\sigma-\beta)^2+t^2} + O(\log q).$$

Choose $\sigma = 1 + \frac{2\delta}{\log q}$. Then $|t| \le \frac{1}{2}(\sigma-1) \le \frac{1}{2}(\sigma-\beta)$

$$\Longrightarrow -\frac{\log q}{2\delta} \le -\frac{8/5}{(\sigma-\beta)} + O(\log q) \le -\frac{8/5 \log q}{3\delta} + O(\log q)$$

(Recall $\beta > 1 - \frac{\delta}{\log(q(|t|+1))} \ge 1 - \frac{\delta}{\log q}$).

This is a contradiction if δ is small enough. ✓

Therefore $t=0$, so ρ must be real.

<u>Uniqueness</u>: Assume there are two zeros

$$1-\frac{\delta}{\log q} \le \beta_1 \le \beta_2 \le 1 \text{ (possibly equal).}$$

Same argument shows $-\frac{1}{\sigma-1} \le -\frac{2\sigma-\beta_1-\beta_2}{(\sigma-\beta_1)(\sigma-\beta_2)} + O(\log q)$

Choose $\sigma = 1+\frac{4\delta}{\log q}$, we obtain the contradiction

$$-\frac{\log q}{4\delta} \le -\frac{8}{25}\frac{\log q}{\delta} + O(\log q). \quad \square$$

Alternative proof of PNT (Sheet 13, ex 2)

We use zero-free region of $\zeta(s)$:

There exists a constant $C>0$ s.t. if $\rho = \sigma + it$ is a non-trivial zero of $\zeta(s)$, then

$$\sigma > 1 - \frac{C}{\log(2+|t|)} .$$

From truncated Perron, for $c = 1 + \frac{1}{\log x}$,

$$\psi(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(-\frac{\zeta'}{\zeta}(s)\right)x^s\frac{ds}{s} + O\left(\frac{x(\log x)^2}{T}\right).$$

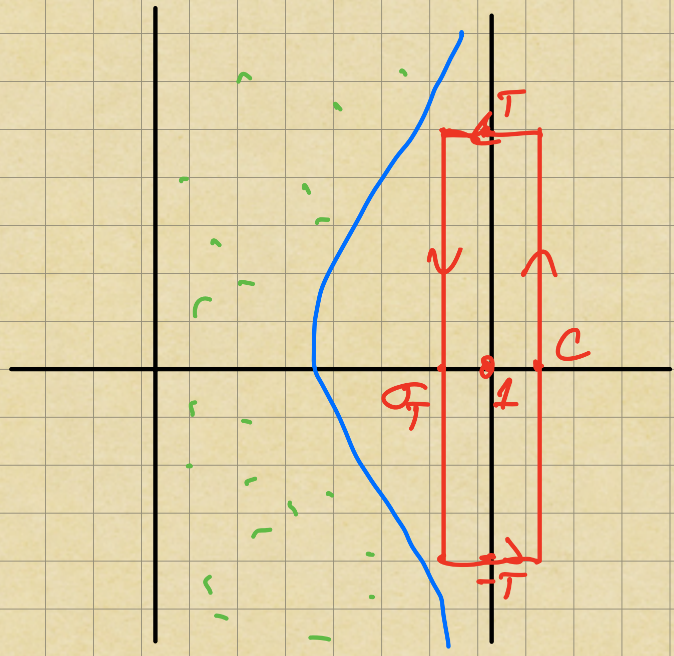
(for $T \le x$)

Let $F(s) = -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}$, integrate along box with corners $c-iT, c+iT, \sigma_T+iT, \sigma_T-iT$, with $\sigma_T = 1 - \frac{C}{2\log T}$.

$\zeta(s)$ has no zeros in this region, so there is only the pole at $s=1$ for $F(s)$ in the box.

Hence $$\frac{1}{2\pi i}\left(\int_{c-iT}^{c+iT}F(s)ds + \int_{c+iT}^{\sigma_T+iT}F(s)ds + \int_{\sigma_T+iT}^{\sigma_T-iT}F(s)ds + \int_{\sigma_T-iT}^{c-iT}F(s)ds\right)$$

$$= \operatorname{Res}_{s=1} F(s) = x.$$



We have that if $s=\sigma+it$ with $|t|\ge 2$
and $\sigma_T \le \sigma \le C$, then

$$-\frac{\zeta'}{\zeta}(\sigma+it) \ll \log(2+|t|)^2$$

(because $|s-\rho| \gg \frac{1}{\log(2+|t|)}$, for any ρ zero of $\zeta(s)$)

Second integral bounded by

$$\int_{C+iT}^{\sigma+iT} \left(-\frac{\zeta'}{\zeta}(s)\right)\cdot\frac{X^s}{s}\,ds \ll (C-\sigma)\cdot\frac{X}{T}\cdot(\log T)^2$$

$$\ll \frac{X}{T}(\log T)^2$$

Similar bound for integral $\underline{IV}$

Integral $\underline{III}$: $$\int_{\sigma+iT}^{\sigma-iT}\left(-\frac{\zeta'}{\zeta}(s)\frac{X^s}{s}\right)ds \ll X^{\sigma_T}(\log T)^3\int_0^T \frac{1}{1+t}\,dt$$

$$\ll X^{1-\frac{c}{2\log T}}\cdot(\log T)^4.$$

Hence $\psi(x) = x + O\left(\frac{x}{T}(\lg x)^2 + x^{1-\frac{c}{2\lg T}} \cdot (\lg T)^4\right)$.

Choose $T = \exp(\sqrt{\lg x})$. ✓

Exam

Syllabus: Material covered in class + exercise sheets

Best practice: Exercise sheets! (last years exam available)

- No Notes allowed!
- Exam will test understanding of tools and techniques, not memorisation on techniqual stuff.
- You will be given a list of results you can use without proof